

# MODELS OF $G$ -SPECTRA AS PRESHEAVES OF SPECTRA

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ABSTRACT. Let  $G$  be a finite group. We give Quillen equivalent models for the category of  $G$ -spectra as categories of presheaves of nonequivariant spectra with explicitly described domain categories. Our preferred model is based on equivariant infinite loop space theory, but the proof that it is a model depends on equivariant Atiyah duality, which leads to another model. Our preferred model recasts equivariant stable homotopy theory in terms of elementary point-set level categories of  $G$ -spans and nonequivariant spectra.

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## INTRODUCTION

A result of Schwede and Shipley [27] asserts that any stable model category  $\mathcal{M}$  is equivalent to a category of enriched presheaves  $\mathcal{S}^{\mathcal{D}}$  with values in a chosen category  $\mathcal{S}$  of spectra. The domain  $\mathcal{S}$ -category  $\mathcal{D}$  is a full  $\mathcal{S}$ -subcategory of  $\mathcal{M}$

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and can be as inexplicit and mysterious as  $\mathcal{M}$  itself. We work through explicit equivalents to the domain category  $\mathcal{D}$  in the case when  $\mathcal{M} = G\mathcal{S}$  is the category of  $G$ -spectra for a finite group  $G$ . We give background in §1, where we explain what holds for general compact Lie groups  $G$ . Restricting to finite groups  $G$ , we describe in §2 how equivariant infinite loop space theory constructs an equivalent model for the relevant  $\mathcal{D}$  starting from quite elementary categories of finite  $G$ -sets. In effect, Theorems 1.2 and 2.3 combine to say that when  $G$  is finite we can study the category  $G\mathcal{S}$  of  $G$ -spectra in terms of the nonequivariant category  $\mathcal{S}$  of spectra and an elementary categorical object  $G\mathcal{E}$  that is described in terms of spans of finite  $G$ -sets. Finally, in §3, we describe how equivariant Atiyah duality gives the real reason that this construction works. A few technical details on the categories we work in are relegated to an appendix.

## 1. BACKGROUND AND MOTIVATION

The equivariant stable homotopy category is of fundamental importance in algebraic topology. It is the natural home in which to study equivariant stable homotopy theory, a subject that has powerful and unexpected nonequivariant applications. For recent examples, it plays a central role in the solution of the Kervaire invariant problem (Hill, Hopkins, and Ravenel [10]), it is central to calculations of topological cyclic homology and therefore to calculations in algebraic K-theory (Hesselholt and Madsen [9], Angeltveit, Gerhardt, and Hesselholt [1] and others), it plays an interesting role by analogy and comparison in motivic homotopy theory (Voevodsky [29, 30]), and it is the motivational starting point for work that develops homological algebra parallelling the topological structure visible in the equivariant stable homotopy category (Kaledin [12]).

Setting up the equivariant stable homotopy category with its attendant model structures takes a fair amount of work. The original version was due to Lewis and May [15], and details of more modern versions have been given by Mandell and May [16, 18, 22]. Most of these sources work with compact Lie groups of equivariance.

Motivated by analogous work of Dugger, Schwede, Shipley, and others, in [7] we gave a general theory describing when enriched model categories are equivalent to categories of enriched presheaves. We start with a good model category  $\mathcal{V}$  in which to enrich things and a  $\mathcal{V}$ -model category  $\mathcal{M}$ . We show that there is often a small  $\mathcal{V}$ -category  $\mathcal{D}$  such that  $\mathcal{M}$  is Quillen equivalent to the category  $\mathcal{V}^{\mathcal{D}}$  of enriched presheaves  $\mathcal{D}^{\text{op}} \rightarrow \mathcal{V}$ . Both conceptually and technically, the model structure on  $\mathcal{V}^{\mathcal{D}}$  is very simple, just being the (projective) model structure induced levelwise from the model structure on  $\mathcal{V}$ . In much of the theory in [7], as in the earlier theory of [27], the category  $\mathcal{D}$  is a full  $\mathcal{V}$ -subcategory of  $\mathcal{M}$  whose objects are generators of  $\mathcal{M}$  in an appropriate sense.

From the point of view of applications and calculations, this is only a starting point. One wants a more concrete understanding of the category  $\mathcal{D}$ . In general, its hom objects  $\mathcal{D}(d, e)$  in  $\mathcal{V}$  and their composition and unit maps

$$(1.1) \quad \mathcal{D}(e, f) \otimes \mathcal{D}(d, e) \longrightarrow \mathcal{D}(d, f) \quad \text{and} \quad \mathbf{I} \longrightarrow \mathcal{D}(d, d),$$

where  $\mathbf{I}$  is the unit object of  $\mathcal{V}$ , may be little easier to understand than for general objects of  $\mathcal{M}$ .

To illustrate the point, we begin by stating two specializations of [7, Theorem 6.7], both of which apply to any compact Lie group  $G$ . Homotopically, they are essentially the same theorem. On the point set level, they are not.

Let  $\mathcal{S}$  be the category of orthogonal spectra and let  $G\mathcal{S}$  be the category of orthogonal  $G$ -spectra, where  $G$  is a compact Lie group. The maps are  $G$ -maps. The category  $G\mathcal{S}$  is closed symmetric monoidal under its smash product, with internal hom objects the function  $G$ -spectra  $F_G(X, Y)$ . It is also enriched over  $\mathcal{S}$ , with hom objects the fixed point spectra  $F_G(X, Y)^G$ . Enriched model categories are discussed in [2, 3, 7, 11, 27] and elsewhere, and  $G\mathcal{S}$  is an  $\mathcal{S}$ -model category under the stable model structure of [18, 19]. The category  $\mathcal{S}$  is a particularly nice enriching category since the sphere orthogonal spectrum  $S$  is cofibrant and  $\mathcal{S}$  satisfies the monoid axiom of [26], by [19, 12.5]. As usual, we write  $X_+$  for the disjoint union of a  $G$ -space and a  $G$ -fixed basepoint.

**Theorem 1.2.** *Let  $G$  be a compact Lie group and let  $G\mathcal{B}$  be the full  $\mathcal{S}$ -subcategory of  $G\mathcal{S}$  whose objects are fibrant approximations of the orbit suspension  $G$ -spectra  $\Sigma_G^\infty(G/H_+)$ , where  $H$  runs over the closed subgroups of  $G$ . Then there is an enriched Quillen adjunction*

$$\mathcal{S}^{G\mathcal{B}} \begin{matrix} \xrightarrow{\mathbb{T}} \\ \xleftarrow{\mathbb{U}} \end{matrix} G\mathcal{S},$$

and it is a Quillen equivalence.

The letter  $\mathcal{B}$  stands for “Burnside”, and  $G\mathcal{B}$  is an enriched version of the Burnside category of  $G$ . When  $G$  is finite, the homotopy category of  $G\mathcal{B}$  has several equivalent algebraic descriptions; we recall one of them in §3.2 (see also [22, §IX.4]).

As something of a joke, but a serious one that is important to the mathematical philosophy of our work, we state a companion theorem. Let  $\mathcal{Z}$  be the category of  $S$ -modules of [4] and let  $G\mathcal{Z}$  be the category of  $S_G$ -modules<sup>1</sup>, details of which are given in [18]. Here  $S_G$  is the sphere  $G$ -spectrum in  $G\mathcal{Z}$ . The category  $G\mathcal{Z}$  is closed symmetric monoidal under its smash product, with internal hom objects the function  $S_G$ -modules  $F_G(X, Y)$ . It is also enriched over  $\mathcal{Z}$ , with hom objects the fixed point  $S$ -modules  $F_G(X, Y)^G$ .

**Theorem 1.3.** *Let  $G$  be a compact Lie group and let  $G\mathcal{B}$  be the full  $\mathcal{Z}$ -subcategory of  $G\mathcal{Z}$  whose objects are cofibrant approximations of the orbit suspension  $G$ -spectra  $\Sigma_G^\infty(G/H_+)$ , where  $H$  runs over the closed subgroups of  $G$ . Then there is an enriched Quillen adjunction*

$$\mathcal{Z}^{G\mathcal{B}} \begin{matrix} \xrightarrow{\mathbb{T}} \\ \xleftarrow{\mathbb{U}} \end{matrix} G\mathcal{Z},$$

and it is a Quillen equivalence.

Now that we have both Theorem 1.2 and Theorem 1.3, we write  $G\mathcal{B}_{\mathcal{S}}$  or  $G\mathcal{B}_{\mathcal{Z}}$  instead of  $G\mathcal{B}$  when it is unclear from context whether we are working in  $G\mathcal{S}$  or  $G\mathcal{Z}$ . Theorem 6.7 of [7] applies to prove both Theorem 1.2 and Theorem 1.3. The orbit  $G$ -spectra give compact generating sets in both  $\mathrm{Ho}(G\mathcal{S})$  and  $\mathrm{Ho}(G\mathcal{Z})$ . We require bifibrant representatives. In Theorem 1.2, the orbit  $G$ -spectra are cofibrant, and fibrant approximation makes them bifibrant. However, we do not have simple enough fibrant approximations to allow easy concrete understanding of these

<sup>1</sup>The notation  $\mathcal{S}$  is short for  $\mathcal{S}\mathcal{S}$  and the notation  $\mathcal{Z}$  is short for  $\mathcal{M}_S$  in the original sources; as a silly mnemonic device,  $\mathcal{Z}$  stands for the  $Z$  in the middle of Elmendorf-Kriz-Mandell-May.

bifibrant orthogonal  $G$ -spectra. In Theorem 1.3, all  $S_G$ -modules are fibrant, and cofibrant approximation makes them bifibrant. Here cofibrant approximation is given by a well understood left adjoint that very nearly preserves smash products, as we explain in the appendix.

Parenthetically, the “goodness” hypothesis of [7, 6.7] holds in Theorem 1.2 because  $\mathcal{S}$  satisfies the monoid axiom, by [18, 7.4]. It holds in Theorem 1.3 by use of the “Cofibration Hypothesis” of [4, p. 146], which also holds equivariantly.

Technically, [7, 6.7] also requires *either* that the unit object of the enriching category  $\mathcal{V}$  be cofibrant *or* that every object in  $\mathcal{V}$  be fibrant. The first hypothesis holds in  $\mathcal{S}$  and the second holds in  $\mathcal{Z}$ . We want to use both of these conditions, and it is impossible to have them both in the same symmetric monoidal model category for the stable homotopy category [14, 24]. Therefore it is no joke that we need both of these results. Theorems 1.2 and 1.3 are related by the following result, which is [18, IV.1.1]; the nonequivariant special case is [18, I.1.1]. In this result,  $G\mathcal{S}$  is given its positive stable model structure from [18] and is denoted  $G\mathcal{S}_+$  to indicate the distinction; in that model structure the sphere  $G$ -spectrum  $S_G$ , like the sphere  $S_G$ -module  $S_G$  in  $G\mathcal{Z}$  is not cofibrant.

**Theorem 1.4.** *For any compact Lie group  $G$ , there is a Quillen equivalence*

$$G\mathcal{S}_+ \begin{matrix} \xrightarrow{\mathbb{N}} \\ \xleftarrow{\mathbb{N}^\#} \end{matrix} G\mathcal{Z}.$$

*The functor  $\mathbb{N}$  is strong symmetric monoidal, hence  $\mathbb{N}^\#$  is lax symmetric monoidal.*

The identity functor is a left Quillen equivalence  $G\mathcal{S}_+ \rightarrow G\mathcal{S}$ . Therefore Theorems 1.2, 1.3, and 1.4, have the following immediate consequence.

**Corollary 1.5.** *The categories  $\mathcal{S}^{G\mathcal{B}_\mathcal{S}}$  and  $\mathcal{Z}^{G\mathcal{B}_\mathcal{Z}}$  are Quillen equivalent. More precisely, there are left Quillen equivalences*

$$\mathcal{S}^{G\mathcal{B}_\mathcal{S}} \rightarrow G\mathcal{S} \leftarrow G\mathcal{S}_+ \rightarrow G\mathcal{Z} \leftarrow \mathcal{Z}^{G\mathcal{B}_\mathcal{Z}}.$$

In fact, we can compare the  $\mathcal{S}$ -category  $G\mathcal{B}_\mathcal{S}$  with the  $\mathcal{Z}$ -category  $G\mathcal{B}_\mathcal{Z}$  via the functor  $\mathbb{N}^\#$ . The adjunction

$$G\mathcal{S}_+ \begin{matrix} \xrightarrow{\mathbb{N}} \\ \xleftarrow{\mathbb{N}^\#} \end{matrix} G\mathcal{Z}$$

is tensored over the adjunction

$$\mathcal{S}_+ \begin{matrix} \xrightarrow{\mathbb{N}} \\ \xleftarrow{\mathbb{N}^\#} \end{matrix} \mathcal{Z}$$

in the sense of [7, 5.5]. We pause to recall what that means. The category  $G\mathcal{S}$  is a bicomplete  $\mathcal{S}$ -category in the sense of [7, §1.3]. In particular,  $G\mathcal{S}$  is tensored over  $\mathcal{S}$ , meaning that we can take the smash product of a  $G$ -spectrum  $X$  with a nonequivariant spectrum  $Y$ . While a direct definition is easy enough, we can also define the tensor  $X \odot Y$  to be  $X \wedge \varepsilon^* Y$ , where  $\varepsilon^*: \mathcal{S} \rightarrow G\mathcal{S}$  is the change of group and universe functor associated to  $\varepsilon: G \rightarrow e$  that assigns a genuine  $G$ -spectrum to a nonequivariant spectrum. The same is true with  $\mathcal{S}$  replaced by  $\mathcal{Z}$ . These functors are discussed in both contexts and are compared in [18]. Results there (see [18, IV.1.1]) imply that

$$\mathbb{N}X \odot \mathbb{N}Y \cong \mathbb{N}(X \odot Y),$$

which is the defining condition for a tensored adjunction. Now [7, 5.24] gives that the  $\mathcal{S}$ -category  $\mathbb{N}^\# G\mathcal{B}_{\mathcal{Z}}$  is “quasi-equivalent” to  $G\mathcal{B}_{\mathcal{S}}$ . Using [7, 4.15, 5.18], this implies a direct proof of the Quillen equivalence of Corollary 1.5. Therefore Theorems 1.2 and 1.3 are equivalent: each implies the other.

We reiterate the generality: the results above hold for *all* compact Lie groups  $G$ . In that generality, we do not know how to simplify the description of the domain category  $G\mathcal{B}$  to transform it into a weakly equivalent  $\mathcal{S}$ -category or  $\mathcal{Z}$ -category that is intuitive and perhaps even familiar, something accessible to study independent of knowledge of the category of  $G$ -spectra that we seek to understand. The purpose of this paper is to obtain such a description when  $G$  is finite and, from here on out,  $G$  will be a fixed finite group.

It will require some work to prove that our more intuitive choices of  $\mathcal{D}$  are weakly  $\mathcal{S}$ -equivalent to  $G\mathcal{B}$  (in the sense of [7, 4.3]) and therefore induce Quillen equivalence between  $\mathcal{S}^{\mathcal{D}}$  and  $\mathcal{S}^{G\mathcal{B}}$  (by [7, 4.4]). That verification is the substance of the paper. We abbreviate weak  $\mathcal{S}$ -equivalence to equivalence henceforward. With fixed object sets, as we will have, it just means an  $\mathcal{S}$ -functor that induces weak equivalences on hom objects in  $\mathcal{S}$ . From now on, we prefer to use the term stable equivalence, rather than weak equivalence, for the weak equivalences in our model categories of spectra and  $G$ -spectra.

In fact, we will obtain two different answers. First, working with orthogonal  $G$ -spectra, we will explain a model for  $G\mathcal{B}$  expressed in terms of the equivariant infinite loop space theory of permutative  $G$ -categories. We will explain the conclusions but leave the relevant details from infinite loop space theory to a sequel [8]. Second, working with  $S$ -modules, we will explain a model for  $G\mathcal{B}$  dictated by close inspection of Atiyah duality for finite  $G$ -sets. Since  $G$  is finite, it makes sense to work with general finite  $G$ -sets rather than just with orbits  $G/H$  throughout.

**Remark 1.6.** We have stated Theorem 1.2 in terms of orbits  $G/H$ . We could equally well shrink the category  $G\mathcal{B}$  by choosing one  $H$  in each conjugacy class. When  $G$  is finite, we can instead expand  $G\mathcal{B}$  to the (skeletally small) full subcategory of  $G\mathcal{S}$  or  $G\mathcal{Z}$  whose objects are bifibrant approximations of the suspension  $G$ -spectra  $\Sigma_G^\infty A_+$ , where  $A$  runs over the nonempty finite  $G$ -sets. By [7, 4.5], [7, 6.7] applies to any set of compact generators, hence all of the theorems above remain true for these expanded versions of the categories  $G\mathcal{B}$ .

Alternatively, we can restrict attention to additive presheaves, namely those that take finite wedges to finite products (which are weakly equivalent to finite wedges). The original categories  $\mathcal{S}^{G\mathcal{B}_{\mathcal{S}}}$  and  $\mathcal{Z}^{G\mathcal{B}_{\mathcal{Z}}}$  are equivalent to the respective categories of additive presheaves defined using finite  $G$ -sets.

Either way, Theorems 1.2 and 1.3 remain valid with  $G\mathcal{B}$  reinterpreted to allow general finite  $G$ -sets rather than just orbits. We can freely switch back and forth between these points of view. Allowing finite  $G$ -sets simplifies notation. We agree to work with this larger category from now on, and we change conventions by letting  $G\mathcal{B}$  denote categories of bifibrant approximations of suspension  $G$ -spectra of finite  $G$ -sets. We can always reinterpret what we do in terms of orbit  $G$ -spectra.

**Remark 1.7.** Cartesian products of  $G$ -sets and  $G$ -spans induce products on the categorical input we shall feed into an infinite loop space machine. We can hope to use this observation as the starting point for a comparison of symmetric monoidal structures on our categories of  $G$ -spectra and presheaves of spectra, but that is

work in progress. Categorical remarks in [7, 2.9, 3.7] indicate the idea and the paragraph above [7, 3.7] makes clear why the comparison is not an obvious one.

As an amusing preamble, we define a simple  $G$ -map  $\varepsilon^2$ , which plays a pivotal role in our work. In fact, its role in Atiyah duality is what makes the entire theory work. The map comes with a simple factorization that plays two key roles. Let  $S^0 = \{*, 1\}$ , where  $*$  is the basepoint.

**Definition 1.8.** For a finite  $G$ -set  $A$ , define based  $G$ -maps

$$\varepsilon: (A \times A)_+ \longrightarrow S^0$$

$$t: (A \times A)_+ \longrightarrow A_+ \quad \text{and} \quad \pi: A_+ \longrightarrow S^0$$

by  $t(a, b) = *$  if  $a \neq b$  and  $t(a, a) = a$ , by  $\pi(a) = 1$ , and by  $\varepsilon = \pi \circ t$ , so that  $\varepsilon(a, b) = *$  if  $a \neq b$  and  $\varepsilon(a, a) = 1$ .

## 2. THE CATEGORY $\mathbb{K}(G\mathcal{E})$ AND THE $G$ -CATEGORY $\mathbb{K}_G(\mathcal{E}_G)$

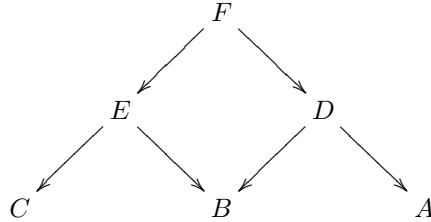
**2.1. The bicategory  $G\mathcal{E}$ .** Our preferred  $\mathcal{S}$ -category  $\mathbb{K}(G\mathcal{E})$  equivalent to  $G\mathcal{B}$  is obtained by applying a suitably well-behaved infinite loop space machine  $\mathbb{K}$  to a simple and concrete “category enriched in permutative categories”  $G\mathcal{E}$  that is defined in terms of finite  $G$ -sets. For the purpose of this paper, we regard  $\mathbb{K}$  as a black box. Recall that a permutative category is a strictly associative and unital small symmetric monoidal category. We can rigidify any small symmetric monoidal category to a permutative category [28], [20, 4.2], and we shall not dwell on the distinction. The notion in quotes does not make mathematical sense since there is no known monoidal structure on the category of permutative categories, but category theory due to the first author [6] explains what these objects are and how to rigidify categories enriched in symmetric monoidal categories to categories enriched in permutative categories. We shall not go into full categorical detail here, but the following definitions describe  $G\mathcal{E}$  informally. The letters  $A$  through  $F$  below all denote finite  $G$ -sets.

**Definition 2.1.** We define a category  $G\mathcal{E}(A)$ . The objects of  $G\mathcal{E}(A)$  are the  $G$ -maps  $p: D \longrightarrow A$ . Its morphisms  $p \longrightarrow q$ ,  $q: E \longrightarrow A$ , are the  $G$ -isomorphisms  $f: D \longrightarrow E$  such that  $q \circ f = p$ . Disjoint union of  $G$ -sets over  $A$  gives  $G\mathcal{E}(A)$  a structure of symmetric monoidal category.

**Definition 2.2.** We define a bicategory  $G\mathcal{E}$  with objects the finite  $G$ -sets and with categories of morphisms between objects specified by  $G\mathcal{E}(A, B) = G\mathcal{E}(B \times A)$ . Think of the objects of  $G\mathcal{E}(A, B)$  as  $G$ -spans  $B \longleftarrow D \longrightarrow A$ . Define composition

$$\circ: G\mathcal{E}(B, C) \times G\mathcal{E}(A, B) \longrightarrow G\mathcal{E}(A, C)$$

via pullbacks, as in the diagram




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<sup>2</sup>This  $\varepsilon$  is an example of the Kronecker  $\delta$  function.

The diagonal maps  $\Delta: B \rightarrow B \times B$  give identities in  $G\mathcal{E}(B, B)$ . Each  $G\mathcal{E}(A, B)$  is a symmetric monoidal category, and the relevant coherence is specified in [6].

**2.2. The  $\mathcal{S}$ -category  $\mathbb{K}(G\mathcal{E})$  and the main theorem.** Technically, we must use an infinite loop space machine  $\mathbb{K}$  that takes values in orthogonal spectra and is the case  $G = e$  of an infinite loop space machine  $\mathbb{K}_G$  that takes values in orthogonal  $G$ -spectra. It must also have a convenient theory of pairings [5, 21], so that we can use it to obtain the composition pairings of  $\mathcal{S}$ -categories. We give details of such a machine in [8]. Applying  $\mathbb{K}$  to  $G\mathcal{E}$ , we obtain an  $\mathcal{S}$ -category  $\mathbb{K}(G\mathcal{E})$ . Retaining the notations of Theorem 1.2, we shall prove the following result, which is the main result of the paper.

**Theorem 2.3.** *Let  $G$  be finite. Then the  $\mathcal{S}$ -category  $\mathbb{K}(G\mathcal{E})$  is equivalent to  $G\mathcal{B}$ . Therefore there is a zigzag of Quillen equivalences between  $\mathcal{S}^{\mathbb{K}(G\mathcal{E})}$  and  $\mathcal{S}^{G\mathcal{B}}$ .*

As already said, we are going to use infinite loop space theory. Implicitly, there is a conceptual issue that must be resolved in order to prove Theorem 2.3 using that tool. There is no known infinite loop space machine that knows about function spectra. That is, given input data  $X$  and  $Y$  (permutative categories,  $E_\infty$ -spaces,  $\Gamma$ -spaces, etc) for an infinite loop space machine  $\mathbb{K}$ , we do not know what input data will have as output the function spectrum  $F(\mathbb{K}X, \mathbb{K}Y)$ . The problem does not make sense as just stated because the output spectra  $\mathbb{K}Z$  are always connective, whereas  $F(\mathbb{K}X, \mathbb{K}Y)$  is generally not. The most that one could hope for in general is to detect the connective cover of  $F(\mathbb{K}X, \mathbb{K}Y)$ . In our case, the relevant function spectra are connective. The conceptual reason for that comes from equivariant Atiyah duality, as we shall see later.

**2.3. Conventions on equivariant categories.** In fact, everything we do depends on first working equivariantly and then passing to fixed points. As a matter of general context, it will help to fix some generic notations, following [7, 18]. Let  $\mathcal{C}$  be any category of objects each of which has an action by our finite group  $G$ .<sup>3</sup> Thus, for each object  $X$  we are given a homomorphism from  $G$  to the automorphism group of  $X$  in  $\mathcal{C}$ . Write  $G\mathcal{C}$  for the category of  $G$ -objects and  $G$ -maps, and write  $\mathcal{C}_G$  for the category of  $G$ -objects and nonequivariant maps. This is a  $G$ -category with  $G$  acting by conjugation on morphism sets, or morphism objects in enriched contexts. The two categories are related conceptually by  $G\mathcal{C} = (\mathcal{C}_G)^G$ . The objects, being  $G$ -objects, are already  $G$ -fixed; we apply the  $G$ -fixed point functor to hom sets or hom objects. This point of view on equivariance in enriched contexts is discussed in more detail and justified conceptually in [18, II§1].

Thus we write  $\mathcal{S}_G$  for the category of orthogonal  $G$ -spectra and non-equivariant maps. The category  $\mathcal{S}_G$  is closed symmetric monoidal and thus enriched over itself, with hom objects the  $G$ -spectra  $F_G(X, Y)$ . To understand  $G\mathcal{S}$  as an  $\mathcal{S}$ -category, we must first understand  $\mathcal{S}_G$  as an  $\mathcal{S}_G$ -category. That is, to understand spectra  $F_G(X, Y)^G$ , we must first understand  $G$ -spectra  $F_G(X, Y)$ . We let  $\mathcal{B}_G$  denote the full  $\mathcal{S}_G$ -subcategory of  $\mathcal{S}_G$  whose objects are those of  $G\mathcal{B}$ . Later we shall use the same notational conventions but with  $\mathcal{S}$  replaced by  $\mathcal{L}$ .

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<sup>3</sup>Some examples below work more generally with an action by  $G$  that translates objects; their full subcategories of  $G$ -fixed objects fit into the framework we are describing.

**2.4. The  $G$ -bicategory  $\mathcal{E}_G$ .** We use similar notations for our categorical input. Again  $A$ ,  $B$ , and  $C$  denote finite  $G$ -sets, but now the  $D$ ,  $E$  and  $F$  of §2.1 are finite sets embedded in a  $G$ -set  $U$  that contains all orbits  $G/H$  countably many times.

**Definition 2.4.** We define a  $G$ -category  $\mathcal{E}_G(A)$ . The objects of  $\mathcal{E}_G(A)$  are the nonequivariant maps  $p: D \rightarrow A$ , where  $A$  is a finite  $G$ -set and  $D$  is a finite subset of  $U$ . The morphisms  $f: p \rightarrow q$ ,  $q: E \rightarrow A$ , are the bijections  $f: D \rightarrow E$  such that  $q \circ f = p$ . The group  $G$  acts on morphisms via the maps  $g: D \rightarrow gD$  and the formula  $(gf)(gd) = gf(d)$ .

**Definition 2.5.** We define a bicategory  $\mathcal{E}_G$  with objects the finite  $G$ -sets and with  $G$ -categories of morphisms between objects specified by  $\mathcal{E}_G(A, B) = \mathcal{E}_G(B \times A)$ . Thinking of the objects of  $G\mathcal{E}(A, B)$  as nonequivariant spans  $B \leftarrow D \rightarrow A$ , composition and identities are defined as in Definition 2.2.

Observe that taking disjoint unions of finite sets over  $A$  will not keep us in  $U$  and is thus not well-defined. Therefore the  $\mathcal{E}_G(A)$  are not symmetric monoidal (let alone permutative)  $G$ -categories in the naive sense of symmetric monoidal categories with  $G$  acting compatibly on all data. In fact, the notion of a genuine permutative  $G$ -category, one that provides input for an equivariant infinite loop space machine, is subtle. We shall give two different solutions to that categorical problem in [8]. In both, genuine permutative  $G$ -categories are described in terms of actions by an  $E_\infty$  operad of  $G$ -categories, to which equivariant infinite loop space theory applies. One of them applies to give each  $\mathcal{E}_G(A)$  such a structure, and the other gives an equivalent way of solving the same problem.

Either way, we can apply an infinite loop space machine  $\mathbb{K}_G$  to obtain  $G$ -spectra  $\mathbb{K}_G\mathcal{E}_G(A)$ . Again we choose our machine to land in the category  $G\mathcal{S}$  of orthogonal  $G$ -spectra, and we require that it take values in the subcategory of positive fibrant orthogonal  $G$ -spectra. We need formal properties of the machine that would not allow us to arrange that it take values in orthogonal  $G$ -spectra that are fibrant in the stable model structure. We shall state the precise properties that we need the  $\mathbb{K}_G\mathcal{E}_G(A)$  to satisfy as we proceed. All omitted proofs are given in [8].

**2.5. Statements of equivariant theorems.** Our starting point is the following two theorems from [8]. We shall say more about them in §2.7. We warn the skeptical reader that the results of this paper depend on Theorems 2.6, 2.7, 2.11, 2.12, and 2.13 from [8]. The knowledgeable expert will immediately accept their plausibility, especially since those of the results which make sense when  $G = e$  have been known for decades. However, the proofs are not obvious and at this writing have not yet been written out in full detail.

**Theorem 2.6** (Equivariant Barratt-Quillen Theorem). *There is a natural stable equivalence*

$$\alpha: \Sigma_G^\infty A_+ \rightarrow \mathbb{K}_G\mathcal{E}_G(A).$$

**Theorem 2.7.** *The composition pairing on spans and the diagonal maps on finite  $G$ -sets induce pairings*

$$\mathbb{K}_G\mathcal{E}_G(B, C) \wedge \mathbb{K}_G\mathcal{E}_G(A, B) \rightarrow \mathbb{K}_G\mathcal{E}_G(A, C)$$

*of  $G$ -spectra and unit maps of  $G$ -spectra*

$$S_G \rightarrow \mathbb{K}_G\mathcal{E}_G(B, B)$$

*that give us a (skeletally) small  $\mathcal{S}_G$ -category  $\mathbb{K}_G\mathcal{E}_G$ .*



It is visible from the definitions that the category  $G\mathcal{E}(A)$  of equivariant spans is equivalent to the  $G$ -fixed point category  $\mathcal{E}_G(A)^G$  and therefore that the category  $G\mathcal{E}(A, B)$  is equivalent to  $\mathcal{E}_G(A, B)^G$ . We show in [8] that the equivariant infinite loop space machine is compatible with the nonequivariant one.

**Theorem 2.8.** *There is a canonical equivalence of  $\mathcal{S}$ -categories*

$$\mathbb{K}(G\mathcal{E}) \longrightarrow (\mathbb{K}_G\mathcal{E}_G)^G.$$

Recall that  $G\mathcal{B}$  is the  $G$ -fixed category of the full  $\mathcal{S}_G$ -subcategory  $\mathcal{B}_G$  of  $G\mathcal{S}$  on bifibrant replacements for the  $G$ -spectra  $\Sigma_G^\infty A_+$ . Recall too that we are free to choose the bifibrant replacements any way we like. By Theorem 2.8, Theorem 2.3 follows by passage to  $G$ -fixed point spectra from its equivariant version.

**Theorem 2.9.** *The  $\mathcal{S}_G$ -category  $\mathbb{K}_G\mathcal{E}_G$  is equivalent to the full  $\mathcal{S}_G$ -category  $\mathcal{B}_G$  of  $G\mathcal{S}$ .*

**2.6. The proof that  $\mathbb{K}_G\mathcal{E}_G$  is equivalent to  $\mathcal{B}_G$ .** To abbreviate notation, write

$$G\mathcal{C} = \mathbb{K}(G\mathcal{E}) \quad \text{and} \quad \mathcal{C}_G = \mathbb{K}_G\mathcal{E}_G.$$

We may think of the  $G$ -spectra  $\mathbb{K}(G\mathcal{E}(A))$  as the objects of  $G\mathcal{C}$  and  $\mathcal{C}_G$ , but we write  $G\mathcal{C}(A, B)$  for the morphism spectra and  $\mathcal{C}_G(A, B)$  for the morphism  $G$ -spectra. We recall the suspension  $G$ -spectrum functor  $\Sigma_G^\infty$  in §4.1, and we abbreviate notation for suspension  $G$ -spectra of finite  $G$ -sets by writing

$$\mathbb{A} = \Sigma_G^\infty A_+, \quad \mathbb{B} = \Sigma_G^\infty B_+, \quad \text{and} \quad \mathbb{C} = \Sigma_G^\infty C_+.$$

Remember that  $\mathcal{C}_G(*, A)$  is the  $G$ -spectrum  $\mathbb{K}_G\mathcal{E}_G(A)$ , which is equivalent to  $\mathbb{A}$  by Theorem 2.6.

*Proof of Theorem 2.9.* We use model categorical arguments, and we must take care as to where to use the stable model structure and where to use the positive stable model structure on  $G\mathcal{S}$ . We use [7, §5.2] to obtain a model structure on the category  $G\mathcal{S}\mathbb{O}\text{-}\mathcal{Cat}$  of  $G\mathcal{S}$ -categories with the same object set  $\mathbb{O}$  as  $\mathcal{E}_G$ . Maps are weak equivalences or fibrations if they induce weak equivalences or fibrations on hom objects in  $G\mathcal{S}$ . Here the nature of the objects is irrelevant; we are concerned with  $G\mathcal{S}$ -categories with one object for each finite  $G$ -set  $A$ . The model structure depends on a prior choice of a model structure on  $G\mathcal{S}$ , and we choose the stable model structure.

Let  $\lambda: Q\mathcal{C}_G \longrightarrow \mathcal{C}_G$  be a cofibrant approximation of  $\mathcal{C}_G$ . By [7, 5.9], since  $S_G$  is cofibrant in the stable model structure each morphism  $G$ -spectrum  $Q\mathcal{C}_G(A, B)$  is cofibrant in  $G\mathcal{S}$ . That would not be true if we started with the positive stable model structure. However, the  $Q\mathcal{C}_G(A, B)$ , like the  $\mathcal{C}_G(A, B)$ , are only fibrant in the positive stable model structure. Let  $\rho: Q\mathcal{C}_G \longrightarrow RQ\mathcal{C}_G$  be a fibrant approximation of  $Q\mathcal{C}_G$ . The morphism  $G$ -spectra  $RQ\mathcal{C}_G(A, B)$  are then bifibrant in the stable model structure. Therefore  $RQ\mathcal{C}_G(*, A)$  is bifibrant for each  $A$ , and it is stably equivalent to  $\mathbb{A}$ . We take the  $RQ\mathcal{C}_G(*, A)$  as the bifibrant approximations of the  $\mathbb{A}$  that we use to define the full  $G\mathcal{S}$ -subcategory  $\mathcal{B}_G$  of  $G\mathcal{S}$ . We also define  $\mathcal{A}_G$  to be the full  $G\mathcal{S}$ -subcategory of  $G\mathcal{S}$  whose objects are the  $Q\mathcal{C}_G(*, A)$ .

By [7, 2.8], we then have canonical  $G\mathcal{S}$ -functors

$$\beta: Q\mathcal{C}_G \longrightarrow \mathcal{A}_G \quad \text{and} \quad \gamma: RQ\mathcal{C}_G \longrightarrow \mathcal{B}_G.$$

The function  $G$ -spectra  $F_G(-, -)$  are the internal homs  $\underline{G\mathcal{S}}(-, -)$ , and the maps

$$\beta: Q\mathcal{C}_G(A, B) \longrightarrow \mathcal{A}_G(A, B) = F_G(Q\mathcal{C}_G(*, A), Q\mathcal{C}_G(*, B))$$

and

$$\gamma: RQ\mathcal{C}_G(A, B) \longrightarrow \mathcal{B}_G(A, B) = F_G(RQ\mathcal{C}_G(*, A), RQ\mathcal{C}_G(*, B))$$

in  $G\mathcal{S}$  are the adjoints of the composition maps

$$\circ: Q\mathcal{C}_G(A, B) \wedge Q\mathcal{C}_G(*, A) \longrightarrow Q\mathcal{C}_G(*, B)$$

and

$$\circ: RQ\mathcal{C}_G(A, B) \wedge RQ\mathcal{C}_G(*, A) \longrightarrow RQ\mathcal{C}_G(*, B).$$

It suffices to prove that each of the maps  $\gamma$  is a stable equivalence. To abbreviate notation, we agree to write

$$Q\mathcal{C}_G(*, A) = Q\mathcal{C}_G A \quad \text{and} \quad RQ\mathcal{C}_G(*, A) = RQ\mathcal{C}_G A.$$

With our notational conventions, it is consistent to write  $Q\mathcal{C}_G(B \times A) = Q\mathcal{C}_G(A, B)$ .

For each finite  $G$ -set  $A$ ,  $\mathbb{A}$  is cofibrant and  $\lambda: Q\mathcal{C}_G A \longrightarrow \mathcal{C}_G A$  is a fibration in the stable model structure. Therefore there is a map  $\mu: \mathbb{A} \longrightarrow Q\mathcal{C}_G A$  such that the following diagram commutes.

$$\begin{array}{ccc} & & Q\mathcal{C}_G A \\ & \nearrow \mu & \downarrow \lambda \\ \mathbb{A} & \xrightarrow{\alpha} & \mathcal{C}_G A \end{array}$$

Since  $\alpha$  and  $\lambda$  are stable equivalences, so is  $\mu$ .

Now we claim that the following diagram of  $G$ -spectra commutes. Here and later we identify  $\mathbb{B} \wedge \mathbb{A} = \Sigma_G^\infty B_+ \wedge \Sigma_G^\infty A_+$  with  $\Sigma_G^\infty (B \times A)_+$  since the two are naturally isomorphic (see §4.1).

$$\begin{array}{ccccc} RQ\mathcal{C}_G(A, B) & \xrightarrow{\gamma} & F_G(RQ\mathcal{C}_G A, RQ\mathcal{C}_G B) & \xrightarrow[F_G(\rho, \text{id})]{\simeq} & F_G(Q\mathcal{C}_G A, RQ\mathcal{C}_G B) \\ \uparrow \rho \simeq & & & \nearrow F_G(\text{id}, \rho) & \searrow F_G(\mu, \text{id}) \\ Q\mathcal{C}_G(A, B) & \xrightarrow{\beta} & F_G(Q\mathcal{C}_G A, Q\mathcal{C}_G B) & & F_G(\mathbb{A}, RQ\mathcal{C}_G B) \\ \uparrow \mu \simeq & & \searrow F_G(\mu, \text{id}) & \nearrow F_G(\text{id}, \rho) & \\ \mathbb{B} \wedge \mathbb{A} & \xrightarrow[\xi]{\simeq} & F_G(\mathbb{A}, \mathbb{B}) & \xrightarrow[F_G(\text{id}, \mu)]{\simeq} & F_G(\mathbb{A}, Q\mathcal{C}_G B) \end{array}$$

The only as yet undefined map in the diagram is  $\xi$ . It is the adjoint of

$$\text{id} \wedge \Sigma_G^\infty \varepsilon: \mathbb{B} \wedge \mathbb{A} \wedge \mathbb{A} \cong \mathbb{B} \wedge \Sigma_G^\infty (A \times A)_+ \longrightarrow \mathbb{B} \wedge \Sigma_G^\infty S^0 = \mathbb{B} \wedge S_G \cong \mathbb{B},$$

where  $\varepsilon: (A \times A)_+ \longrightarrow S^0$  is the  $G$ -map specified in Definition 1.8. We shall see later (at the end of §3.4) that Atiyah duality implies that  $\xi$  is a stable equivalence. The maps  $\mu$  and  $\rho$  are also stable equivalences. The maps  $F_G(\rho, \text{id})$  and  $F_G(\mu, \text{id})$  that are labeled  $\simeq$  are stable equivalences by [7, 4.13] since  $\rho$  and  $\mu$  are maps between cofibrant objects and  $RQ\mathcal{C}_G B$  is fibrant. The maps  $F_G(\text{id}, \mu)$  and  $F_G(\text{id}, \rho)$  that are labeled  $\simeq$  are stable equivalences by [18, III.3.9], which shows that the functor  $F_G(\mathbb{A}, -)$  preserves stable equivalences. Granting our claim that the diagram commutes, it follows that  $\gamma$  is a stable equivalence since all other outer arrows of the diagram are stable equivalences.

To prove that the diagram commutes, we consider its adjoint. Remembering that  $\lambda \circ \mu = \alpha$ , we see that the adjoint can be written in the following form.

$$\begin{array}{ccccccc}
RQ\mathcal{C}_G(A, B) \wedge \mathbb{A} & \xrightarrow{\text{id} \wedge \mu} & RQ\mathcal{C}_G(A, B) \wedge Q\mathcal{C}_G A & \xrightarrow{\text{id} \wedge \rho} & RQ\mathcal{C}_G(A, B) \wedge RQ\mathcal{C}_G(A) & & \\
\uparrow \rho \wedge \text{id} & \nearrow \rho \wedge \mu & \uparrow \rho \wedge \text{id} & \nearrow \rho \wedge \rho & \searrow \circ & & \\
Q\mathcal{C}_G(A, B) \wedge \mathbb{A} & \xrightarrow{\text{id} \wedge \mu} & Q\mathcal{C}_G(A, B) \wedge Q\mathcal{C}_G A & \xrightarrow{\circ} & Q\mathcal{C}_G B & \xrightarrow{\rho} & RQ\mathcal{C}_G B \\
\uparrow \mu \wedge \text{id} & \nearrow \mu \wedge \mu & \downarrow \lambda \wedge \lambda & & \downarrow \lambda & \nearrow \mu & \\
\mathbb{B} \wedge \mathbb{A} \wedge \mathbb{A} & \xrightarrow{\alpha \wedge \alpha} & \mathcal{C}_G(A, B) \wedge \mathcal{C}_G A & \xrightarrow{\circ} & \mathcal{C}_G B & \xleftarrow{\alpha} & \mathbb{B} \\
& & & \xrightarrow{\varepsilon} & & & 
\end{array}$$

Since  $\lambda$  and  $\rho$  are maps of  $G\mathcal{S}$ -categories, it is apparent that all parts of the diagram commute except possibly for the bottom trapezoid. We explain its commutativity in a generalized form in the following subsection.  $\square$

**2.7. The key commutative diagram.** Taking  $(A, B, C) = (*, A, B)$ , the following result gives the commutative diagram that is needed to complete the proof of Theorem 2.9. We again write  $\mathbb{A} = \Sigma_G^\infty A_+$ , but otherwise we revert to the notations of §2.5. Remember that  $\mathbb{B} \wedge \mathbb{A}$  can be identified with  $\Sigma_G^\infty(B \times A)_+$ .

**Theorem 2.10.** *The following diagram of  $G$ -spectra commutes.*

$$\begin{array}{ccc}
\mathbb{C} \wedge \mathbb{B} \wedge \mathbb{B} \wedge \mathbb{A} & \xrightarrow{\alpha \wedge \alpha} & \mathbb{K}_G \mathcal{E}_G(C \times B) \wedge \mathbb{K}_G \mathcal{E}_G(B \times A) \\
\downarrow \text{id} \wedge \varepsilon \wedge \text{id} & & \downarrow \circ \\
\mathbb{C} \wedge \mathbb{A} & \xrightarrow{\alpha} & \mathbb{K}_G \mathcal{E}_G(C \times A)
\end{array}$$

The proof is based on three naturality properties of the equivalence

$$\alpha: \Sigma_G^\infty A_+ \longrightarrow \mathbb{K}_G \mathcal{E}_G(A)$$

that will be proven in [8]. First, taking cartesian products of spaces over  $A$  and  $B$  induces a pairing  $\mathcal{E}_G(A) \times \mathcal{E}_G(B) \longrightarrow \mathcal{E}_G(A \times B)$ . In turn, that induces a pairing of  $G$ -spectra

$$\wedge: \mathbb{K}_G \mathcal{E}_G(A) \wedge \mathbb{K}_G \mathcal{E}_G(B) \longrightarrow \mathbb{K}_G \mathcal{E}_G(A \times B).$$

The map  $\alpha$  commutes with smash products in the following sense.

**Theorem 2.11.** *The following diagram of  $G$ -spectra commutes.*

$$\begin{array}{ccc}
\Sigma_G^\infty A_+ \wedge \Sigma_G^\infty B_+ & \xrightarrow{\alpha \wedge \alpha} & \mathbb{K}_G \mathcal{E}_G(A) \wedge \mathbb{K}_G \mathcal{E}_G(B) \\
\downarrow \cong & & \downarrow \wedge \\
\Sigma_G^\infty(A \times B)_+ & \xrightarrow{\alpha} & \mathbb{K}_G \mathcal{E}_G(A \times B)
\end{array}$$

Second,  $\alpha$  is natural with respect to  $G$ -maps  $f: A \longrightarrow B$ . Post-composition of maps over  $A$  with  $f$  induces a functor  $f_!: \mathcal{E}_G(A) \longrightarrow \mathcal{E}_G(B)$ . In turn, that induces a map of  $G$ -spectra

$$\mathbb{K}_G f_!: \mathbb{K}_G \mathcal{E}_G(A) \longrightarrow \mathbb{K}_G \mathcal{E}_G(B).$$

**Theorem 2.12.** *The following diagram of  $G$ -spectra commutes.*

$$\begin{array}{ccc} \Sigma_G^\infty A_+ & \xrightarrow{\alpha} & \mathbb{K}_G \mathcal{E}_G(A) \\ \Sigma_G^\infty f_+ \downarrow & & \downarrow \mathbb{K}_G f_! \\ \Sigma_G^\infty B_+ & \xrightarrow{\alpha} & \mathbb{K}_G \mathcal{E}_G(B) \end{array}$$

Third,  $\alpha$  enjoys a kind of contravariant naturality with respect to inclusions  $i: A \rightarrow B$  of finite  $G$ -sets. Restriction of maps over  $B$  to maps over  $A$  induces a functor  $i^*: \mathcal{E}_G(B) \rightarrow \mathcal{E}_G(A)$ . In turn, that induces a map of  $G$ -spectra

$$\mathbb{K}_G i^*: \mathbb{K}_G \mathcal{E}_G(B) \rightarrow \mathbb{K}_G \mathcal{E}_G(A).$$

Define a based  $G$ -map  $t: B_+ \rightarrow A_+$  by  $ti(a) = a$  and  $t(b) = *$  if  $b \notin \text{im}(i)$ .

**Theorem 2.13.** *The following diagram of  $G$ -spectra commutes.*

$$\begin{array}{ccc} \Sigma_G^\infty B_+ & \xrightarrow{\alpha} & \mathbb{K}_G \mathcal{E}_G(B) \\ \Sigma^\infty t \downarrow & & \downarrow \mathbb{K}_G i^* \\ \Sigma_G^\infty A_+ & \xrightarrow{\alpha} & \mathbb{K}_G \mathcal{E}_G(A) \end{array}$$

*Proof of Theorem 2.10.* We expand the required diagram. We have the inclusion

$$\text{id} \times \Delta \times \text{id}: C \times B \times A \rightarrow C \times B \times B \times A$$

and we let

$$t: (C \times B \times B \times A)_+ \rightarrow (C \times B \times A)_+$$

be the corresponding based  $G$ -map. We let

$$\pi: C \times B \times A \rightarrow C \times A$$

be the projection. As observed in Definition 1.8,  $\varepsilon: (B \times B)_+ \rightarrow S^0$  is the composite of the map  $t: (B \times B)_+ \rightarrow B_+$  determined by the inclusion  $\Delta: B \rightarrow B \times B$  and the projection  $\pi: B_+ \rightarrow \{1\}_+ = S^0$ . Analogously, we observe that the functor

$$\circ: \mathcal{E}_G(C \times B \times B \times A) \rightarrow \mathcal{E}_G(C \times A)$$

is the composite  $\pi_! \circ (\text{id} \times \Delta \times \text{id})^*$ . The left vertical composite in the following diagram is  $\text{id} \wedge \varepsilon \wedge \text{id}$  and, by definition, the right vertical composite is the composition  $\circ$  of Theorem 2.7. The diagram commutes by the previous three theorems.

$$\begin{array}{ccc} \Sigma_G^\infty (C \times B)_+ \wedge \Sigma_G^\infty (B \times A)_+ & \xrightarrow{\alpha \wedge \alpha} & \mathbb{K}_G \mathcal{E}_G(C \times B) \wedge \mathbb{K}_G \mathcal{E}_G(B \times A) \\ \downarrow \wedge & & \downarrow \wedge \\ \Sigma^\infty (C \times B \times B \times A)_+ & \xrightarrow{\alpha} & \mathbb{K}_G \mathcal{E}_G(C \times B \times B \times A) \\ \Sigma_G^\infty t \downarrow & & \downarrow \mathbb{K}_G (\text{id} \times \Delta \times \text{id})^* \\ \Sigma^\infty (C \times B \times A)_+ & \xrightarrow{\alpha} & \mathbb{K}_G \mathcal{E}_G(C \times B \times A) \\ \Sigma^\infty \pi \downarrow & & \downarrow \mathbb{K}_G \pi_! \\ \Sigma_G^\infty (C \times A)_+ & \xrightarrow{\alpha} & \mathbb{K}_G \mathcal{E}_G(C \times A) \end{array}$$

□

### 3. ATIYAH DUALITY FOR FINITE $G$ -SETS

**3.1. The self-duality of the homotopy category  $\mathbf{Ho}G\mathcal{B}$ .** Before starting work on what we need, we recall an old result that illuminates our context. Let  $\mathbf{Ho}G\mathcal{B}$  be the homotopy category of  $G\mathcal{B}$ . It is the full subcategory of the homotopy category  $\mathbf{Ho}G\mathcal{S}$  or its equivalent  $\mathbf{Ho}G\mathcal{Z}$  with objects the  $G$ -spectra  $\mathbb{A}$  as  $A$  runs over the nonempty finite  $G$ -sets.

The category  $[G\mathcal{E}]$  of  $G$ -spans is obtained from the bicategory  $G\mathcal{E}$  of  $G$ -spans by identifying spans  $p$  and  $q$  from  $A$  to  $B$  if there is an isomorphism between them. Composition by pullbacks then becomes strictly associative and unital. We add spans  $A \rightarrow B$  by taking disjoint unions. That gives the morphism set  $[G\mathcal{E}](A, B)$  a structure of abelian monoid. We apply the Grothendieck construction to obtain an abelian group of morphisms  $A \rightarrow B$ . Elementary verifications show that this construction gives an additive category  $\mathcal{A}b[G\mathcal{E}]$ . Thus we start with spans of finite  $G$ -sets and build in additive structure in the most naive possible fashion. The following result is [15, V.9.6].

**Theorem 3.1.** *The categories  $\mathbf{Ho}G\mathcal{B}$  and  $\mathcal{A}b[G\mathcal{E}]$  are isomorphic.*

Therefore the category  $\mathbf{Ho}G\mathcal{B}$  is self-dual, in the sense that it is isomorphic to its opposite category. Indeed a span  $B \leftarrow D \rightarrow A$  can equally well be viewed as a span  $A \leftarrow D \rightarrow B$ . As we shall see, Atiyah duality for smooth  $G$ -manifolds specializes to give a direct homotopical proof that  $\mathbf{Ho}G\mathcal{B}$  is self-dual.

Duality in a general symmetric monoidal category  $\mathcal{V}$  with unit object  $\mathbf{I}$  is described categorically in [15, III§1] and [23], for example. Two objects  $X$  and  $Y$  are dual if there are maps  $\eta: \mathbf{I} \rightarrow X \otimes Y$  and  $\varepsilon: Y \otimes X \rightarrow \mathbf{I}$  satisfying the standard “triangle identities” under composition. If  $\mathcal{V}$  is closed, the adjoint of  $\varepsilon$  is then an isomorphism from  $Y$  to the categorical dual  $DX = \underline{\mathcal{V}}(X, \mathbf{I})$ . The category  $G\mathcal{E}$  of finite  $G$ -sets is cartesian monoidal, with unit the one-point  $G$ -set  $*$ , and the self-duality of the finite  $G$ -set  $A$  is given by the equivalence classes  $\eta = [\pi, \Delta]$  and  $\varepsilon = [\Delta, \pi]$  of the spans

$$(3.2) \quad * \xleftarrow{\pi} A \xrightarrow{\Delta} A \times A \quad \text{and} \quad A \times A \xleftarrow{\Delta} A \xrightarrow{\pi} *.$$

Atiyah duality lifts this structure to the homotopy category of  $G$ -spectra. For that, we can use any good model for the stable homotopy category. However, to lift this structure on the point-set level and still retain homotopically meaningful objects, it is convenient to work in a category in which every object is fibrant and cofibrant approximation is given by an explicit functor that is well-behaved with respect to smash products.

The category  $G\mathcal{Z}$  of  $S_G$ -modules has these properties. Working in it, we shall show that the self-duality just described very nearly lifts to give a point-set level self-dual category of suspension  $G$ -spectra  $\mathbb{A}$ . In fact, we can lift composition precisely, but we cannot quite lift units. The only result from this theory needed to complete the proof of Theorem 2.9 is the mere homotopical fact that the map  $\xi$  used there is an equivalence, which drops out of our more precise discussion at the end of §3.4.

**3.2. The categories  $G\mathcal{Z}$ ,  $G\mathcal{B}$ , and  $\mathcal{B}_G$ .** We work in  $G\mathcal{Z}$  in this section. We shall give relevant background in §4.3, and we just give the bare minimum of notation here. Like  $G\mathcal{S}$ ,  $G\mathcal{Z}$  is closed symmetric monoidal under its smash product with internal hom objects denoted  $F_G(X, Y)$ . The construction of  $G\mathcal{Z}$  starts from

the Lewis-May category  $G\mathcal{S}p$  of  $G$ -spectra, and  $S_G$ -modules are  $G$ -spectra with additional structure. We use  $G\mathcal{S}p$  as a convenient half-way house between the category  $G\mathcal{T}$  of based  $G$ -spaces and the category  $G\mathcal{Z}$ .

As we recall in §4.3, we have an elementary suspension  $G$ -spectrum functor

$$\Sigma_G^\infty : G\mathcal{T} \longrightarrow G\mathcal{S}p.$$

There is a left adjoint  $\mathbb{F} : G\mathcal{S}p \longrightarrow G\mathcal{Z}$ , also recalled in §4.3, and we define

$$\Sigma_G^\infty : G\mathcal{T} \longrightarrow G\mathcal{Z}$$

to be the composite  $\mathbb{F} \circ \Sigma_G^\infty$ . Suspension  $G$ -spectra have natural structures as  $S_G$ -modules, and there is a natural stable equivalence of  $S_G$ -modules

$$\gamma : \Sigma_G^\infty X \longrightarrow \Sigma_G^\infty X.$$

Viewing  $\Sigma_G^\infty$  as a functor  $G\mathcal{T} \longrightarrow G\mathcal{Z}$ , it is strong symmetric monoidal. However, the  $\Sigma_G^\infty X$  are not cofibrant. The functor  $\Sigma_G^\infty$  takes based  $G$ -CW complexes  $X$ , such as  $A_+$  for a finite  $G$ -set  $A$ , to cofibrant  $S_G$ -modules. Then  $\Sigma_G^\infty$  may be viewed as a cofibrant replacement functor for  $\Sigma_G^\infty$ . In particular, we write  $\mathbf{S}_G^\infty = \Sigma_G^\infty S^0$  and have a cofibrant approximation  $\gamma : \mathbf{S}_G^\infty \longrightarrow S_G$  of the unit object  $S_G$ .

As before, we consider finite  $G$ -sets  $A$ ,  $B$ , and  $C$ , but we now agree to write

$$\mathbb{A} = \Sigma_G^\infty A_+, \quad \mathbb{B} = \Sigma_G^\infty B_+, \quad \text{and} \quad \mathbb{C} = \Sigma_G^\infty C_+.$$

The  $\mathbb{A}$  are bifibrant objects of  $G\mathcal{Z}$  and we let  $G\mathcal{B}$  and  $\mathcal{B}_G$  be the full subcategories of  $G\mathcal{Z}$  and  $\mathcal{Z}_G$  whose objects are the  $S_G$ -modules  $\mathbb{A}$ , where  $A$  runs over the nonempty finite  $G$ -sets. Thus  $\mathcal{B}_G$  is enriched in  $G\mathcal{Z}$  and  $G\mathcal{B} = (\mathcal{B}_G)^G$  is enriched in the category  $\mathcal{Z}$  of  $S$ -modules.

As we show in §4.3, the functor  $\Sigma_G^\infty$  is almost strong symmetric monoidal. Precisely, there is a natural isomorphism

$$(3.3) \quad \mathbb{A} \wedge \mathbb{B} \cong \mathbf{S}_G^\infty \wedge \Sigma_G^\infty (A \times B)_+$$

with appropriate coherence properties with respect to associativity and commutativity. Since  $S_G$  is the unit for the smash product, we can compose with

$$\gamma \wedge \text{id} : \mathbf{S}_G^\infty \wedge \Sigma_G^\infty (A \times B)_+ \longrightarrow \Sigma_G^\infty (A \wedge B)_+$$

to give a pairing as if  $\Sigma_G^\infty$  were a lax symmetric monoidal functor. However, the map  $\gamma : \mathbf{S}_G^\infty \longrightarrow S_G$  points the wrong way for the unit map of such a functor.

**Remark 3.4.** We shall use that  $G\mathcal{S}p$  and  $G\mathcal{Z}$  are tensored over  $G\mathcal{T}$  and that  $\Sigma^\infty$ ,  $\mathbb{F}$ , and therefore  $\Sigma_G^\infty$  commute with tensors. Tensors in  $G\mathcal{T}$  are just smash products. We shall write  $Y \odot X$  for the tensor of a  $G$ -spectrum or  $S_G$ -module  $Y$  with a based  $G$ -space  $X$ . The usual notation is  $\wedge$ , which can be confusing. Then  $\Sigma_G^\infty X$  is isomorphic to  $S_G \odot X$  and therefore  $\Sigma_G^\infty X$  is isomorphic to  $\mathbf{S}_G^\infty \odot X$ . We shall say a bit more about tensors and use them to prove (3.3) in §4.3.

**3.3. The self-duality of the category  $G\mathcal{B}$ .** We consider duality theory in  $\text{Ho}G\mathcal{Z}$ , but the theory can be transported to or from any other model for the homotopy category. The most complete exposition is given in [15], which works in the homotopy category  $\text{Ho}G\mathcal{S}p$  of  $G$ -spectra. Specializing the general theory of duality in symmetric monoidal categories, for  $S_G$ -modules  $X$  and  $Y$  we have a natural map

$$(3.5) \quad \zeta : Y \wedge DX = Y \wedge F_G(X, S_G) \longrightarrow F_G(X, Y)$$

in  $G\mathcal{Z}$ , namely the adjoint of

$$\mathrm{id} \wedge \varepsilon: Y \wedge DX \wedge X \longrightarrow Y \wedge S_G \cong Y,$$

where  $\varepsilon$  is the evaluation map. The map  $\zeta$  induces an isomorphism in  $\mathrm{Ho}\mathcal{Z}$  when  $X$  or  $Y$  is dualizable.

The  $G$ -spectra  $\mathbb{A}$  are self-dual. That is, we have maps  $\varepsilon: \mathbb{A} \wedge \mathbb{A} \longrightarrow S_G$  whose adjoints  $\tilde{\varepsilon}: \mathbb{A} \longrightarrow D\mathbb{A}$  are isomorphisms in  $\mathrm{Ho}G\mathcal{Z}$ . Thus we have isomorphisms

$$(3.6) \quad \delta: \mathbb{B} \wedge \mathbb{A} \xrightarrow{\mathrm{id} \wedge \tilde{\varepsilon}} \mathbb{B} \wedge D\mathbb{A} \xrightarrow{\zeta} F_G(\mathbb{A}, \mathbb{B}).$$

Therefore, composition and units

$$(3.7) \quad F_G(\mathbb{B}, \mathbb{C}) \wedge F_G(\mathbb{A}, \mathbb{B}) \longrightarrow F_G(\mathbb{A}, \mathbb{C}) \quad \text{and} \quad S_G \longrightarrow F_G(\mathbb{B}, \mathbb{B})$$

can be expressed as maps

$$(3.8) \quad \mathbb{C} \wedge \mathbb{B} \wedge \mathbb{B} \wedge \mathbb{A} \longrightarrow \mathbb{C} \wedge \mathbb{A} \quad \text{and} \quad S_G \longrightarrow \mathbb{B} \wedge \mathbb{B}$$

in  $\mathrm{Ho}G\mathcal{Z}$ . We have already seen just such a composition, in Theorem 2.10. There we were working in  $G\mathcal{S}$  and had not yet considered duality. We now want to understand these maps in  $G\mathcal{Z}$  and to connect them to duality. We will need to take the isomorphisms (3.3) and the cofibrant approximation  $\gamma: \mathbf{S}_G \longrightarrow S_G$  into account since we are working in  $G\mathcal{Z}$ . We would not be able to obtain strict units in any context. The notion of a weakly unital enriched category was introduced in [7, §5.6] to formalize what we see, and we summarize the constructions to follow.

**Construction 3.9.** We shall construct a weakly unital  $G\mathcal{Z}$ -category  $\mathcal{D}_G$ . Its objects are the  $S_G$ -modules  $\mathbb{A}$  for finite  $G$ -sets  $A$ . Its morphism  $S_G$ -modules are specified by  $\mathcal{D}_G(\mathbb{A}, \mathbb{B}) = \mathbb{B} \wedge \mathbb{A}$ . Its composition maps

$$(3.10) \quad \mathbb{C} \wedge \mathbb{B} \wedge \mathbb{B} \wedge \mathbb{A} \longrightarrow \mathbb{C} \wedge \mathbb{A}$$

in  $G\mathcal{Z}$  are specified in Theorem 3.19 below. Its weak unital property is expressed by maps  $\eta_A: \mathbf{S}_G^\infty \wedge \mathbf{S}_G^\infty \longrightarrow \mathbb{A} \wedge \mathbb{A}$  and stable equivalences  $\xi_A: \mathbb{A} \longrightarrow \mathbb{A}$  displayed in (3.16) of Definition 3.13 and (3.24) of Definition 3.21. We shall express  $\delta = \zeta(\mathrm{id} \wedge \tilde{\varepsilon})$  as a map from the weakly unital  $G\mathcal{Z}$ -category  $\mathcal{D}_G$  to the unital  $G\mathcal{Z}$ -category  $\mathcal{B}_G$ .

**3.4. Space level Atiyah duality for finite  $G$ -sets.** To construct  $\mathcal{D}_G$ , we need representative maps in  $G\mathcal{Z}$  for the equivalences  $\zeta$  and  $\tilde{\varepsilon}$  that appear in (3.6). We have seen that the definition of  $\zeta$  is formal, coming from general theory that applies to any closed symmetric monoidal category, including both  $G\mathcal{Z}$  and  $\mathrm{Ho}G\mathcal{Z}$ . For a finite  $G$ -set  $A$ , the self-duality of  $\mathbb{A}$  is determined by appropriate maps

$$\eta: S_G \longrightarrow \mathbb{A} \wedge \mathbb{A} \quad \text{and} \quad \varepsilon: \mathbb{A} \wedge \mathbb{A} \longrightarrow S_G$$

in  $\mathrm{Ho}G\mathcal{Z}$ , and we need representative maps for them in  $G\mathcal{Z}$ . The map  $\varepsilon$  is induced from the elementary map  $\varepsilon$  of Definition 1.8, but the observation that it plays a key role in Atiyah duality seems not to have been previously noticed;  $\eta$  on the other hand requires desuspension by representation spheres in its definition.

The following pair of definitions specify  $\varepsilon$  and  $\eta$  in  $G\mathcal{Z}$ , starting from space level  $G$ -maps. In both,  $A$  is a finite  $G$ -set. In the second,  $V = \mathbb{R}[A]$  is the real representation generated by  $A$  with its standard inner product, so that  $|a| = 1$  for  $a \in A$ . Since we are working on the space level, we may view  $A_+ \wedge S^V$  as the wedge over  $a \in A$  of the spaces (not  $G$ -spaces)  $\{a\}_+ \wedge S^V$ , with  $G$  acting by  $g(a, v) = (ga, gv)$ . There is no such wedge decomposition after passage to  $G$ -spectra.

**Definition 3.11.** Recall that  $\varepsilon = \varepsilon_A: (A \times A)_+ \longrightarrow S^0$  is the  $G$ -map given by  $\varepsilon(a, b) = *$  if  $a \neq b$  and  $\varepsilon(a, a) = 1$ . Recall too that  $(A \times B)_+$  can be identified with  $A_+ \wedge B_+$  and that the functor  $\Sigma_G^\infty$  is almost strong symmetric monoidal. We shall also write  $\varepsilon$  for the composite map of  $S_G$ -modules

$$(3.12) \quad \mathbb{A} \wedge \mathbb{A} \cong \mathbf{S}_G^\infty \wedge \Sigma_G^\infty(A \times A)_+ \xrightarrow{\text{id} \wedge \Sigma_G^\infty \varepsilon} \mathbf{S}_G^\infty \wedge \mathbf{S}_G^\infty \xrightarrow{\gamma \wedge \gamma} S_G \wedge S_G \cong S_G.$$

**Definition 3.13.** Embed  $A$  as the basis of the real representation  $V = \mathbb{R}[A]$ . The normal bundle of the embedding is just  $A \times V$ , and its Thom complex is  $A_+ \wedge S^V$ . We obtain an explicit tubular embedding  $\nu: A \times V \longrightarrow V$  by setting

$$\nu(a, v) = a + (\rho(|v|)/|v|)v,$$

where  $\rho: [0, \infty) \longrightarrow [0, d)$  is a homeomorphism for some  $d < 1/2$ ;  $\nu$  is a  $G$ -map since  $|gv| = |v|$  for all  $g$  and  $v$ . Applying the Pontryagin-Thom construction, we obtain a  $G$ -map  $t: S^V \longrightarrow A_+ \wedge S^V$ , which is an equivariant pinch map

$$S^V \longrightarrow \bigvee_{a \in A} S^V \cong A_+ \wedge S^V.$$

To be more precise, after collapsing the complement of the tubular embedding to a point, we use  $\nu^{-1}$  to expand each small homeomorphic copy of  $S^V$  to the canonical full-sized one; explicitly, if  $|w| < d$ , then

$$\nu^{-1}(a + w) = (a, (\rho^{-1}(|w|)/|w|)w).$$

The diagonal map on  $A$  induces the Thom diagonal  $\Delta: A_+ \wedge S^V \longrightarrow A_+ \wedge A_+ \wedge S^V$ , and we let

$$(3.14) \quad \eta_A: S^V \longrightarrow A_+ \wedge A_+ \wedge S^V$$

be the composite  $\Delta \circ t$ . Explicitly,

$$(3.15) \quad \eta_A(v) = \begin{cases} (a, a, (\rho^{-1}(|w|)/|w|)w) & \text{if } v = a + w \text{ where } a \in A \text{ and } |w| < d \\ * & \text{otherwise.} \end{cases}$$

The negative sphere  $G$ -spectrum  $S^{-V}$  in  $G\mathcal{S}p$  is obtained by applying the left adjoint of the  $V^{th}$ -space functor to  $S^0$ , and  $S_G$  is isomorphic to  $S^V \odot S^{-V}$  (see [15, I.4.3] and [18, IV.2.2]). Taking the tensor of  $\eta_A$  with  $S^{-V}$  we obtain a map of  $G$ -spectra

$$S_G \cong S^V \odot S^{-V} \longrightarrow (A_+ \wedge A_+ \wedge S^V) \odot S^{-V} \cong (A_+ \wedge A_+) \odot S_G \cong \Sigma_G^\infty(A_+ \wedge A_+).$$

Applying the functor  $\mathbb{F}$  to this map and smashing with  $\mathbf{S}_G^\infty$  we obtain the second map in the diagram

$$(3.16) \quad S_G \cong S_G \wedge S_G \xleftarrow{\gamma \wedge \gamma} \mathbf{S}_G^\infty \wedge \mathbf{S}_G^\infty \xrightarrow{\eta_A} \mathbf{S}_G^\infty \wedge \Sigma_G^\infty(A \times A)_+ \cong \mathbb{A} \wedge \mathbb{A}.$$

The following result is a reminder about space level Atiyah duality. The notion of a  $V$ -duality was defined and explained for smooth  $G$ -manifolds in [15, §III.5].

**Proposition 3.17.** *The maps*

$$\eta_A: S^V \longrightarrow A_+ \wedge A_+ \wedge S^V \quad \text{and} \quad \varepsilon_A \wedge \text{id}: A_+ \wedge A_+ \wedge S^V \longrightarrow S^V$$

*specify a  $V$ -duality between  $A_+$  and itself.*



*Proof.* This could be proven from scratch by proving the required triangle identities, but in fact it is a special case of equivariant Atiyah duality for smooth  $G$ -manifolds,  $A$  being a 0-dimensional example. Our specification of  $\eta_A$  is a specialization of the description of  $\eta_M$  for a general smooth  $G$ -manifold  $M$  given in [15, p. 152]. We claim that  $\varepsilon_A \wedge \text{id}$  is a specialization of the description of  $\varepsilon_M$  for a general smooth  $G$ -manifold given there. Indeed, letting  $s$  be the zero section of the normal bundle  $\nu$  of the embedding  $A \subset \mathbb{R}[A] = V$ , we have the composite embedding

$$A \xrightarrow{\Delta} A \times A \xrightarrow{s \times \text{id}} (A \times V) \times A \cong A \times A \times V.$$

The normal bundle of this embedding is  $A \times V$ , and we may view

$$\Delta \times \text{id}: A \times V \longrightarrow A \times A \times V$$

as giving a big tubular neighborhood. The Pontryagin-Thom map here is obtained by smashing the map  $t: (A \times A)_+ \longrightarrow A_+$  that sends  $(a, b)$  to  $a$  if  $a = b$  and to  $*$  if  $a \neq b$  with the identity map of  $S^V$ . Composing with the map induced by the projection  $\pi: A_+ \longrightarrow S^0$  that sends  $a$  to 1, this gives  $\varepsilon \wedge \text{id}$ . We first pointed out this factorization of  $\varepsilon$  in Definition 1.8, and we have used it before, in the proof of Theorem 2.10.  $\square$

Tensoring with  $S^{-V}$ , applying the functor  $\mathbf{S}_G^\infty \wedge \mathbb{F}$ , and composing with  $\gamma$ , we obtain the explicit duality maps in  $G\mathcal{Z}$  displayed in (3.12) and (3.16).

**Corollary 3.18.** *The adjoint of the map  $\varepsilon: \mathbb{A} \wedge \mathbb{A} \longrightarrow S_G$  in (3.12) is a stable equivalence  $\tilde{\varepsilon}: \mathbb{A} \longrightarrow D\mathbb{A}$  in  $G\mathcal{Z}$ .*

For any finite  $G$ -sets  $A$  and  $B$ , the adjoint  $\xi$  of

$$\text{id} \wedge \varepsilon: \mathbb{B} \wedge \mathbb{A} \wedge \mathbb{A} \longrightarrow \mathbb{B} \wedge S_G \cong \mathbb{B}$$

is the composite of equivalences

$$\mathbb{B} \wedge \mathbb{A} \xrightarrow{\text{id} \wedge \tilde{\varepsilon}} \mathbb{B} \wedge D\mathbb{A} \xrightarrow{\zeta} F_G(\mathbb{A}, \mathbb{B}).$$

We can transport this isomorphism in  $\text{Ho}G\mathcal{Z}$  to the equivalent category  $\text{Ho}G\mathcal{S}$ , and this finally completes the proof that the map  $\xi$  used in the proof of Theorem 2.9 in §2.6 is an equivalence.

**3.5. The composition and weak unit maps.** We compare the categories  $\mathcal{B}_G$  and  $\mathcal{D}_G$  here, discussing composition and weak unit maps separately. We have described representative maps in  $G\mathcal{Z}$  for the comparison isomorphism (3.6) in  $\text{Ho}G\mathcal{Z}$ . The composition and unit maps of  $\mathcal{B}_G$  in (3.7) are of course represented by the corresponding composition and unit maps in  $G\mathcal{Z}$ . To complete the definition of  $\mathcal{D}_G$  promised in Construction 3.9, we must describe composition and unit maps in  $G\mathcal{Z}$  that represent the composition and unit maps (3.8) in  $\text{Ho}G\mathcal{Z}$ .

**Theorem 3.19.** *Under the described equivalence in  $G\mathcal{Z}$  that represents the isomorphism (3.6) in  $\text{Ho}G\mathcal{Z}$ , the composition map (3.7) in  $G\mathcal{Z}$  is equivalent to the composition map (3.10) that is obtained from the map of finite  $G$ -sets*

$$\text{id} \wedge \varepsilon \wedge \text{id}: C_+ \wedge (B \times B)_+ \wedge A_+ \longrightarrow C_+ \wedge A_+$$

*by passage to  $S_G$ -modules using the almost symmetric monoidal property (3.3) of the functor  $\Sigma_G^\infty$ .*

*Proof.* We have the adjoint  $\tilde{\varepsilon}: \mathbb{B} \rightarrow D\mathbb{B}$  and also the counit  $\varepsilon: D\mathbb{B} \wedge \mathbb{B} \rightarrow S_G$  of the  $(\wedge, F_G)$  adjunction in  $G\mathcal{Z}$ . Formal properties of adjunctions give the following commutative diagram in  $G\mathcal{Z}$ .

$$\begin{array}{ccc}
 \mathbb{C} \wedge \mathbb{B} \wedge \mathbb{B} \wedge \mathbb{A} & \xrightarrow{\Sigma_G^\infty(\text{id} \wedge \varepsilon \wedge \text{id})} & \mathbb{C} \wedge \mathbb{A} \\
 \text{id} \wedge \tilde{\varepsilon} \wedge \text{id} \wedge \tilde{\varepsilon} \downarrow & & \downarrow \text{id} \wedge \tilde{\varepsilon} \\
 \mathbb{C} \wedge D\mathbb{B} \wedge \mathbb{B} \wedge D\mathbb{A} & \xrightarrow{\text{id} \wedge \varepsilon \wedge \text{id}} & \mathbb{C} \wedge D\mathbb{A} \\
 \zeta \wedge \zeta \downarrow & & \downarrow \zeta \\
 F_G(\mathbb{B}, \mathbb{C}) \wedge F_G(\mathbb{A}, \mathbb{B}) & \xrightarrow{\circ} & F_G(\mathbb{A}, \mathbb{C})
 \end{array}$$

At the bottom, we do not know that the function  $S_G$ -modules or their smash product are cofibrant, but all objects at the top are cofibrant and thus bifibrant. In general, to compute the smash product of  $G$ -spectra  $X$  and  $Y$  in the homotopy category, we should take the smash product of cofibrant approximations  $QX$  and  $QY$  of  $X$  and  $Y$ . Since all objects of  $G\mathcal{Z}$  are fibrant, to compute a map  $X \wedge Y \rightarrow Z$  in the homotopy category, we should represent it by a map  $QX \wedge QY \rightarrow QZ$  and take its homotopy class. As the diagram displays, that is what we are doing.  $\square$

The unit  $S_G \rightarrow F_G(\mathbb{A}, \mathbb{A})$  is represented by the (formal) composite

$$(3.20) \quad S_G \xrightarrow{\eta} \mathbb{A} \wedge \mathbb{A} \xrightarrow{\text{id} \wedge \tilde{\varepsilon}} \mathbb{A} \wedge D\mathbb{A} \xrightarrow{\zeta} F_G(\mathbb{A}, \mathbb{A}).$$

that is obtained by inverting the map  $\gamma \wedge \gamma$  in (3.16) to obtain the map denoted  $\eta$ . This statement only asserts that the displayed map in  $\text{Ho}G\mathcal{Z}$  gives a unit there, and that much is a formal consequence of the definition of dualizability [15, III.1.1]. We need a precise interpretation in  $G\mathcal{Z}$ .

**Definition 3.21.** For each  $a \in A$ , define a map  $\xi_a: \{a\}_+ \wedge S^V \rightarrow \{a\}_+ \wedge S^V$  by

$$(3.22) \quad \xi_a(a, v) = \begin{cases} (a, (\rho^{-1}(|w|)/|w|)w) & \text{if } v = a + w \text{ and } |w| < d \\ * & \text{otherwise,} \end{cases}$$

where  $\rho$  is as in Definition 3.13. Then the wedge of the  $\xi_a$  is a  $G$ -map

$$(3.23) \quad \xi_A: A_+ \wedge S^V \rightarrow A_+ \wedge S^V;$$

$\xi_A$  is  $G$ -homotopic to the identity map of  $A_+ \wedge S^V$  via the explicit  $G$ -homotopy

$$h(a, v, t) = \begin{cases} (a, v) & \text{if } t = 0 \text{ or } v = a \\ (a, (1-t)v + t(\rho^{-1}(t|w|)/|w|)w) & \text{if } v = a + w \text{ and } t|w| < d \\ * & \text{otherwise.} \end{cases}$$

With  $\eta_A$  as specified in (3.14), easy and perhaps illuminating inspections show that the following unit diagrams already commute in  $G\mathcal{T}$ , before passage to homotopy. In both,  $A$  and  $B$  are finite  $G$ -sets. In the first,  $V = \mathbb{R}[A]$ . In the second,  $V = \mathbb{R}[B]$  and we move  $S^V$  from the right to the left for clarity.

$$\begin{array}{ccc}
 B_+ \wedge A_+ \wedge S^V & \xrightarrow{\text{id}_{B_+} \wedge \text{id}_{A_+} \wedge \eta_A} & B_+ \wedge A_+ \wedge A_+ \wedge A_+ \wedge S^V \\
 \text{id}_{B_+} \wedge \xi_A \downarrow & & \swarrow \text{id}_{B_+} \wedge \varepsilon \wedge \text{id}_{A_+ \wedge S^V} \\
 B_+ \wedge A_+ \wedge S^V & & 
 \end{array}$$

$$\begin{array}{ccc}
S^V \wedge B_+ \wedge A_+ & \xrightarrow{\eta_B \wedge \text{id}_{B_+ \wedge A_+}} & S^V \wedge B_+ \wedge B_+ \wedge B_+ \wedge A_+ \\
\xi_B \wedge \text{id}_{A_+} \downarrow & \swarrow \text{id}_{S^V \wedge B_+} \wedge \varepsilon \wedge \text{id}_{A_+} & \\
S^V \wedge B_+ \wedge A_+ & & 
\end{array}$$

Tensoring with  $S^{-V}$  and recalling the natural isomorphisms

$$(X \wedge S^V) \odot S^{-V} \cong X \odot S_G \cong \Sigma_G^\infty X$$

for based  $G$ -spaces  $X$ , we see that the space level  $G$ -equivalence  $\xi_A$  induces a spectrum level  $G$ -equivalence

$$(3.24) \quad \xi_A: \mathbb{A} \longrightarrow \mathbb{A}.$$

Tensoring with  $S^{-V}$  and using (3.3) to pass to smash products of  $S_G$ -modules, a little diagram chase shows that the previous pair of diagrams in  $G\mathcal{T}$  gives rise to the following pair of commutative diagrams in  $G\mathcal{Z}$ . These express the unit laws for a weakly unital  $G\mathcal{Z}$ -category  $\mathcal{D}_G$  [7, §5.6] with objects the  $\mathbb{A}$  and composition as specified in Theorem 3.19. Before displaying the diagrams, we insert notation that facilitates the comparison with [7, §5.6].

**Remark 3.25.** The unit laws in [7, §5.6] allow us to start with any chosen cofibrant approximation  $\gamma: QS_G \rightarrow S_G$  of the unit  $S_G$ . We are led by (3.16) to choose our cofibrant approximation to be  $\gamma \wedge \gamma: \mathbf{S}_G^\infty \wedge \mathbf{S}_G^\infty \rightarrow S_G \wedge S_G \cong S_G$ . We agree to use the notation  $\gamma: QS_G \rightarrow S_G$  for this map.

$$\begin{array}{ccc}
\mathbb{B} \wedge \mathbb{A} \wedge QS_G & \xrightarrow{\text{id} \wedge \eta_A} & \mathbb{B} \wedge \mathbb{A} \wedge \mathbb{A} \wedge \mathbb{A} \\
\text{id} \wedge \xi_A \wedge \gamma \downarrow & & \downarrow \circ \\
\mathbb{B} \wedge \mathbb{A} \wedge S_G & \xrightarrow{\cong} & \mathbb{B} \wedge \mathbb{A}
\end{array}$$
  

$$\begin{array}{ccc}
QS_G \wedge \mathbb{B} \wedge \mathbb{A} & \xrightarrow{\eta_B \wedge \text{id}} & \mathbb{B} \wedge \mathbb{B} \wedge \mathbb{B} \wedge \mathbb{A} \\
\gamma \wedge \xi_B \wedge \text{id} \downarrow & & \downarrow \circ \\
S_G \wedge \mathbb{B} \wedge \mathbb{A} & \xrightarrow{\cong} & \mathbb{B} \wedge \mathbb{A}
\end{array}$$

Taking  $A = S^0$  in our second space level diagram and changing  $B$  to  $A$ , we also obtain the following commutative diagram in  $G\mathcal{Z}$ . It expresses the unit condition required of the map  $\delta: \mathcal{D}_G \rightarrow \mathcal{B}_G$ .

$$\begin{array}{ccc}
QS_G \wedge \mathbb{A} & \xrightarrow{\eta_A \wedge \text{id}} & \mathbb{A} \wedge \mathbb{A} \wedge \mathbb{A} \\
\gamma \wedge \xi_A \downarrow & & \downarrow \text{id} \wedge \varepsilon \\
S_G \wedge \mathbb{A} & \xrightarrow{\cong} & \mathbb{A}
\end{array}$$

Passing to adjoints, this gives the commutative diagram

$$\begin{array}{ccc}
 QS_G & \xrightarrow{\eta_A} & \mathbb{A} \wedge \mathbb{A} \\
 \gamma \downarrow & & \downarrow \text{id} \wedge \tilde{\varepsilon} \\
 S_G & & \mathbb{A} \wedge D\mathbb{A} \\
 \eta \downarrow & & \downarrow \zeta \\
 F_G(\mathbb{A}, \mathbb{A}) & \xrightarrow{F_G(\xi_A, \text{id})} & F_G(\mathbb{A}, \mathbb{A}),
 \end{array}$$

where  $\eta$  is adjoint to the identity map of  $\mathbb{A}$ .

**3.6. The category of presheaves with domain  $G\mathcal{D}$ .** Passing to  $G$ -fixed points from the weakly unital  $G\mathcal{L}$ -category  $\mathcal{D}_G$  and the  $G\mathcal{L}$ -map  $\delta: \mathcal{D}_G \rightarrow \mathcal{B}_G$ , we obtain a weakly unital  $\mathcal{L}$ -category  $G\mathcal{D}$  and a map  $\delta: G\mathcal{D} \rightarrow G\mathcal{B}$  of weakly unital  $\mathcal{L}$ -categories. Weakly unital presheaves and presheaf categories are defined in [7, 5.26]. By [7, 5.27], we obtain the same category of presheaves  $\mathcal{X}^{G\mathcal{B}}$  using unital or weakly unital presheaves. Since  $\delta$  is an equivalence, we can adapt the methodology of [7, §4] to prove the following result. However, since we find the use of weakly unital categories unpleasant and our main result Theorem 2.3 more satisfactory, we shall leave the details to the interested reader. Nevertheless, it is this equivalence that best captures the geometric intuition behind our results.

**Theorem 3.26.** *The categories  $\mathcal{X}^{G\mathcal{D}}$  and  $\mathcal{X}^{G\mathcal{B}}$  are Quillen equivalent.*

#### 4. APPENDIX: SUSPENSION SPECTRA AND SMASH PRODUCTS

**4.1. Suspension spectra and smash products in  $G\mathcal{S}$ .** We have used the notation  $\Sigma_G^\infty$  for the suspension  $G$ -spectrum functor from the category  $G\mathcal{T}$  of based  $G$ -spaces to the category  $G\mathcal{S}$  of orthogonal  $G$ -spectra. For an inner product space  $V$  and a based  $G$ -space  $X$ , the  $V^{th}$  space of  $\Sigma_G^\infty X$  is just  $X \wedge S^V$ . This functor, which is also denoted by  $F_0$ , is the left adjoint of the zero<sup>th</sup> space  $(-)_0: G\mathcal{S} \rightarrow G\mathcal{T}$ . Nonequivariantly, it is part of [19, 1.8] that for based spaces  $X$  and  $Y$ ,  $F_0X \wedge F_0Y$  is naturally isomorphic to  $F_0(X \wedge Y)$ . The categorical proof of that result in [19, §21] applies equally well equivariantly and gives the following result.

**Proposition 4.1.** *The functor  $\Sigma_G^\infty: G\mathcal{T} \rightarrow G\mathcal{S}$  is strong symmetric monoidal.*

Therefore the zero<sup>th</sup> space functor is lax symmetric monoidal, but of course that functor is not homotopically meaningful except on objects that are fibrant in the stable model structure. There is no known fibrant replacement functor in that model structure that is well-behaved with respect to smash products.

Nonequivariantly, a homotopically meaningful version of the adjunction  $(\Sigma^\infty, \Omega^\infty)$  has been worked out for symmetric spectra by Sagave and Schlichtkrull [25] and for symmetric and orthogonal spectra by Lind [17], who compares his constructions with the adjunction  $(\Sigma^\infty, \Omega^\infty)$  in  $\mathcal{S}p$  (see §4.3 below) and with its analogue for  $\mathcal{X}$ . This work generalizes to the equivariant context, although details have not been written down.

**4.2. A lax monoidal fibrant replacement functor in  $G\mathcal{S}$ .** Parenthetically, the following observation has long been understood but is not written down in the literature. It is an immediate consequence of Theorem 1.4.

**Proposition 4.2.** *For any compact Lie group  $G$ , the unit  $\eta: E \rightarrow \mathbb{N}^\# \mathbb{N}E$  of the adjunction between  $G\mathcal{S}$  and  $G\mathcal{Z}$  specifies a lax monoidal fibrant replacement functor for the positive stable model structure on  $G\mathcal{S}$ .*

**Remark 4.3.** Nonequivariantly, Kro [13] has given a different lax monoidal positive fibrant replacement functor for orthogonal spectra. As he notes, his construction does not apply to symmetric spectra. However, by [19, 3.3], the unit  $\eta: E \rightarrow \mathbb{N}^\# \mathbb{U}PE$  of the composite of the adjunction  $(\mathbb{P}, \mathbb{U})$  between symmetric and orthogonal spectra and the adjunction  $(\mathbb{N}, \mathbb{N}^\#)$  gives a lax monoidal positive fibrant replacement functor for symmetric spectra.

**4.3. Suspension spectra and smash products in  $G\mathcal{Z}$ .** We sketch the relationships among  $G\mathcal{S}$ ,  $G\mathcal{Sp}$ , and  $G\mathcal{Z}$  that we have used. For more information, see [22, XXIV], [18, §IV.2], and the nonequivariant precursor [4].

We have a category  $G\mathcal{P}$  of (coordinate-free)-prespectra. Its objects  $Y$  are based  $G$ -spaces  $Y(V)$  and based  $G$ -maps  $Y(V) \wedge S^W \rightarrow Y(W - V)$  for  $V \subset W$ . Here  $V$  and  $W$  are sub inner product spaces of a complete  $G$ -universe  $U$ . A  $G$ -spectrum is a  $G$ -prespectrum  $Y$  whose adjoint  $G$ -maps  $Y(V) \rightarrow \Omega^{W-V} Y(W)$  are homeomorphisms. The suspension  $G$ -prespectrum functor  $\Pi$  sends a based  $G$ -space  $X$  to  $\{X \wedge S^V\}$ . There is a left adjoint spectrification functor  $L: G\mathcal{P} \rightarrow G\mathcal{Sp}$ , and the suspension  $G$ -spectrum functor  $\Sigma_G^\infty: G\mathcal{S} \rightarrow G\mathcal{Sp}$  is  $L \circ \Pi$ . Explicitly, let

$$Q_G X = \operatorname{colim} \Omega^V \Sigma^V X,$$

where  $V$  runs over the finite dimensional subspaces of a complete  $G$ -universe  $U$ . Then the  $V^{\text{th}}$   $G$ -space of  $\Sigma_G^\infty X$  is  $Q_G \Sigma^V X$ .

All objects of  $G\mathcal{Sp}$  are fibrant, and the zero<sup>th</sup> space functor  $\Omega_G^\infty: G\mathcal{Sp} \rightarrow G\mathcal{S}$  is now homotopically meaningful. For a based  $G$ -CW complex  $X$  (with based attaching maps),  $\Sigma_G^\infty X$  is cofibrant in  $G\mathcal{Sp}$ . In particular, the sphere  $G$ -spectrum  $S_G = \Sigma_G^\infty S^0$  is cofibrant. However,  $G\mathcal{Sp}$  is not symmetric monoidal under the smash product. The implicit trade offs just described are intrinsic to the mathematics, as was explained by Lewis [14]; see [24] for a more recent discussion.

We summarize some constructions in [4] that work in exactly the same fashion equivariantly as nonequivariantly. Starting with the complete  $G$ -universe  $U$ , we have the  $G$ -space  $\mathcal{L}(j)$  of linear isometries  $U^j \rightarrow U$ , with  $G$  acting by conjugation. These spaces form an  $E_\infty$   $G$ -operad. The  $G$ -monoid  $\mathcal{L}(1)$  gives rise to a monad  $\mathbb{L}$  on  $G\mathcal{Sp}$ . Its algebras are called  $\mathbb{L}$ -spectra, and we have the category  $G\mathcal{Sp}[\mathbb{L}]$  of  $\mathbb{L}$ -spectra. It has a smash product  $\wedge_{\mathcal{L}}$  which is associative and commutative but not unital. The action map  $\xi: \mathbb{L}Y \rightarrow Y$  of an  $\mathbb{L}$ -spectrum  $Y$  is a stable equivalence.

Suspension  $G$ -spectra are naturally  $\mathbb{L}$ -spectra. In particular, the sphere  $G$ -spectrum  $S_G = \Sigma_G^\infty S^0$  is an  $\mathbb{L}$ -spectrum. For  $\mathbb{L}$ -spectra  $Y$ , there is a natural stable equivalence  $\lambda: S_G \wedge_{\mathcal{L}} Y \rightarrow Y$ . The  $S_G$ -modules are those  $Y$  for which  $\lambda$  is an isomorphism, and they are the objects of  $G\mathcal{Z}$ . All suspension  $G$ -spectra are  $S_G$ -modules, and so are all  $\mathbb{L}$ -spectra of the form  $S_G \wedge_{\mathcal{L}} Y$ . The smash product  $\wedge$  on  $S_G$ -modules is just the restriction of the smash product  $\wedge_{\mathcal{L}}$ , and it gives  $G\mathcal{Z}$  its symmetric monoidal structure.

We have a sequence of Quillen left adjoints

$$G\mathcal{T} \xrightarrow{\Sigma_G^\infty} G\mathcal{S}p \xrightarrow{\mathbb{L}} G\mathcal{S}p[\mathbb{L}] \xrightarrow{\mathbb{J}} G\mathcal{Z},$$

where  $\mathbb{L}X$  is the free  $\mathbb{L}$ -spectrum generated by a  $G$ -spectrum  $X$  and  $\mathbb{J}Y = S_G \wedge_{\mathcal{L}} Y$  is the  $S_G$ -module generated by an  $\mathbb{L}$ -spectrum  $Y$ . We let  $\mathbb{F} = \mathbb{J}\mathbb{L}$ ; then  $\mathbb{L}$ ,  $\mathbb{J}$ , and  $\mathbb{F}$  are Quillen equivalences. The composite  $\gamma = \xi \circ \lambda: \mathbb{F}Y \rightarrow Y$  is a stable equivalence for any  $\mathbb{L}$ -spectrum  $Y$ . We have defined  $\Sigma_G^\infty$  to be the composite functor  $\mathbb{F}\Sigma_G^\infty$ , and we have the natural stable equivalence of  $S_G$ -modules  $\gamma: \Sigma_G^\infty X \rightarrow \Sigma_G^\infty X$ .

The tensor  $Y \odot X$  of a  $G$ -prespectrum and a based  $G$ -space  $X$  has  $V^{th}$   $G$ -space  $Y(V) \wedge X$ . When  $Y$  is a  $G$ -spectrum, the  $G$ -spectrum  $Y \odot X$  is  $L(\ell Y \odot X)$ , where  $\ell Y$  is the underlying  $G$ -prespectrum of  $Y$  [15, I.3.1]. Tensors in  $G\mathcal{S}p[\mathbb{L}]$  and  $G\mathcal{Z}$  are inherited from those in  $G\mathcal{S}p$ . All of our left adjoints are enriched over  $\mathcal{T}$  and preserve tensors. This leads to the following relationship between  $\wedge$  and  $\Sigma_G^\infty$ .

**Proposition 4.4.** *For based  $G$ -spaces  $X$  and  $Y$ , there are natural isomorphisms*

$$\Sigma_G^\infty X \wedge \Sigma_G^\infty Y \cong (\mathbf{S}_G^\infty \wedge \mathbf{S}_G^\infty) \odot (X \wedge Y) \cong \mathbf{S}_G^\infty \wedge \Sigma_G^\infty (X \wedge Y).$$

*Proof.* We have  $\Sigma^\infty X \cong S_G \odot X$  and therefore

$$\Sigma_G^\infty X = \mathbb{F}\Sigma^\infty X \cong \mathbb{F}(S_G \odot X) \cong \mathbb{F}S_G \odot X = \mathbf{S}_G^\infty \odot X.$$

We also have

$$(\mathbf{S}_G^\infty \odot X) \wedge (\mathbf{S}_G^\infty \odot Y) \cong (\mathbf{S}_G^\infty \wedge \mathbf{S}_G^\infty) \odot (X \wedge Y)$$

and the conclusion follows.  $\square$

Taking  $X = A_+$  and  $Y = B_+$ , this gives the isomorphism (3.3) used in §3.

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